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Note

The Evaluation of Oscillatory Integrals with Infinite Limits

In a wide number of applications in applied mathematics the problem arises of evaluating integrals of the type

$$\int_0^\infty f(x) \begin{matrix}\cos\\ \sin\end{matrix} kx\, dx, \tag{1}$$

where k is a constant. In certain circumstances, k may take large values and considerable difficulty is then experienced in computing the integral by conventional methods, due to the extremely strong cancellation of the positive and negative contributions from the rapidly oscillatory integrand.

The conventional methods of approximating $f(x)$ by a low order polynomial, as proposed, for example, by Filon [1], Clendenin [2], and Flinn [3], do not apply here because of the infinite range. An alternative approach, adopted by Hurwitz and Zweifel [4], Hurwitz, Pfeifer, and Zweifel [5], Saenger [6], and Balbine and Franklin [7], is to subdivide the range and to integrate between the successive zeros of $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} kx$, thus, converting the infinite integral to a summation. The main objection to this approach is that the resulting series may converge slowly. An attempt to remedy this defect was made by Longman [8], who used a variation of Euler's transformation to accelerate convergence.

In the present work the possibility of using the more general nonlinear transformation of Shanks [9] is investigated.

The range of integration is subdivided in accordance with the half-cycles of the integrand into the subranges $[a_n, a_{n+1}]$, $n = 0, 1, 2, \ldots$, where in the case of the integrals (1), a_n is given by

$$a_n = n\pi/k. \tag{2}$$

The rather more general oscillatory integrals

$$\int_0^\infty f(x) \begin{matrix}\cos\\ \sin\end{matrix} kx^2\, dx, \tag{3}$$

which arise in many applications are also treated here. For such integrals a_n is given by

$$a_n = (n\pi/k)^{1/2}. \tag{4}$$

In both cases a low order Gauss–Legendre quadrature formula [10] is employed to carry out the integrations over each half-cycle $[a_n, a_{n+1}]$ according to the prescription

$$\int_a^b g(x)\,dx = \tfrac{1}{2}(b-a)\sum_{i=1}^{p} w_i g[\tfrac{1}{2}(b+a) + \tfrac{1}{2}(b-a)\,x_i] + E_p\,. \tag{5}$$

This result represents the basic p-point Gauss–Legendre quadrature formula, E_p being the associated error, and the weights w_i and abscissas x_i are extensively tabulated by Stroud and Secrest [11]. To minimize the contributions from truncation and round-off errors the intervals $[a_n, a_{n+1}]$ were again subdivided uniformly and formula (5) was applied successively in each of the subintervals to yield the values

$$T_n = \int_{a_n}^{a_{n+1}} g(x)\,dx, \tag{6}$$

where $g(x)$ represents the appropriate integrand from integrals (1) or (3). The most widely applied combination in practice was to use six 2-point rules in the interval $[a_n, a_{n+1}]$, although on some occasions twelve 2-point rules were employed for greater accuracy.

The actual integrals required are, of course, given by

$$A = \sum_{i=0}^{\infty} T_i\,, \tag{7}$$

and the next section describes the application of the Shanks' technique for accelerating the convergence of the partial sums

$$A_n = \sum_{i=0}^{n} T_i\,. \tag{8}$$

Shanks' transformation involves the use of the operators e_j which transform a given sequence $\{A_n\}$, $n = 0, 1, 2,\ldots$, into the sequence $\{B_{j,n}\}$, $n = j, j+1, j+2,\ldots$, according to

$$\{B_{j,n}\} = e_j\{A_n\}. \tag{9}$$

The general term in $\{B_{j,n}\}$ is given as the ratio of two determinants of order $(n+1)$ in Shanks' paper where details of restrictions imposed and conditions to be satisfied in the use of the operators are specified. The most frequently used transformation is e_1 which produces the particularly simple result

$$B_{1,n} = (A_{n+1}A_{n-1} - A_n{}^2)(A_{n+1} + A_{n-1} - 2A_n)^{-1}. \tag{10}$$

The first application concerns the evaluation of the integral

$$I_1(\alpha, k) = \int_0^\infty e^{-\alpha x} \sin kx \, dx = k(\alpha^2 + k^2)^{-1}. \tag{11}$$

In this case the terms of the sequence $\{A_n\}$ are easily evaluated analytically, yielding

$$A_n = k(\alpha^2 + k^2)^{-1}\{1 - (-1)^{n+1} \exp[-(n+1)\alpha\pi/k]\}. \tag{12}$$

The geometric convergence of this sequence implies *immediate* convergence of the sequence $e_1\{A_n\}$ to the exact limiting value, that is $B_{1,n} = k(\alpha^2 + k^2)^{-1}$ for *all n*. On the other hand, it is easily shown that the well known Euler transformation [8] produces a sequence which converges less quickly than even the original sequence $\{A_n\}$, for values of α which are greater than the critical value $(k \ln 3)/\pi$. The Euler transformation does, in fact, produce accelerated convergence for α less than the critical value, but is, of course, always inferior to the immediately converging Shanks' transformation. The implication of this result is that the Shanks' transformation is extremely powerful in dealing with integrals where T_n exhibits predominantly exponential decay.

As a practical example of this class of integral, consideration is given to

$$I_2 = \int_0^\infty x^{-1} \exp(-x/2) \sin x \, dx = \tan^{-1} 2 = 1.107149\ldots. \tag{13}$$

The transformation e_1 was applied iteratively to the sequence $\{A_n\}$ to give the sequences $e_1\{A_n\}$, $e_1^2\{A_n\}$,..., the first few members of which are shown in Table I. For comparison purposes, Euler's transformation was applied to the terms T_n to produce the sequence of partial sums $\{E_n\}$.

TABLE I

Successive Sequences for Evaluating I_2

n	T_n	A_n	E_n	$e_1\{A_n\}$	$e_1^2\{A\}$
0	1.148148	1.148148	0.574074		
1	−0.045820	1.102328	0.849656	1.107254	
2	0.005519	1.107847	0.982409	1.107141	1.107149
3	−0.000809	1.107038	1.046562	1.107150	
4	0.000130	1.107163	1.077652		

The half-cycle contributions, T_n, were evaluated over $[n\pi, (n+1)\pi]$ using twelve 2-point Gauss–Legendre formulas. It will be noted that the Euler sequence

$\{E_n\}$ converges less well than the original sequence $\{A_n\}$. In contrast $e_1\{A_n\}$ converges extremely rapidly and $e_1^2\{A_n\}$ converges immediately to the limiting value. Further, the transformation e_2 could also be applied to $\{A_n\}$ and again yields the limiting value immediately, namely $B_{2,2} = 1.107149...$. A further example involving a different family of integrals is

$$I_3 = \int_\pi^\infty x^{-2} \sin x \, dx = -\text{ci}(\pi) = -0.073668..., \tag{14}$$

where ci is the cosine integral as defined by Gradshteyn and Ryzhik [12]. The results are demonstrated in Table II.

TABLE II

Successive Sequences for Evaluating I_3

n	T_n	A_n	E_n	$e_1\{A_n\}$	$e_1^2\{A_n\}$	$e_1^3\{A_n\}$
0	−9.6230	−9.6230	−4.8115			
1	3.3180	−6.3049	−6.3877	−7.3496		
2	−1.6737	−7.9786	−7.5529	−7.3744	−7.3677	
3	1.0078	−6.9708	−7.4305	−7.3633	−7.3667	−7.3669
4	−0.6730	−7.6439	−7.3901	−7.3689	−7.3670	
5	0.4812	−7.1626	−7.3614	−7.3657		
6	−0.3612	−7.5238	−7.3651			
	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$

The half-cycle contributions, T_n, were evaluated over $[(n+1)\pi, (n+2)\pi]$ using six 2-point Gauss–Legendre formulas.

It will be observed once more that the Shanks' transformations give rise to sequences which converge more rapidly than that produced by the Euler method.

As a final example, the integral

$$I_4 = \int_0^\infty x^2 \sin 100x^2 \, dx, \tag{15}$$

which converges in the mean only (in the Abel sense) is considered. The exact result is

$$I_4 = (\pi/2)^{1/2}/4000 = 3.133285... \times 10^{-4}, \tag{16}$$

which is readily obtained by standard integration, using the integrating factor $\exp(-\gamma x^2)$ as γ tends to zero. The numerical results are shown in Table III.

TABLE III

Successive Sequences for Evaluating I_4

n	T_n	A_n	E_n	$e_1\{A_n\}$	$e_1^2\{A_n\}$	$e_1^3\{A_n\}$	$e_1^4\{A_n\}$
0	1.2177	1.2177	6.0883				
1	−2.1650	−0.9474	3.7199	2.7356			
2	2.7998	1.8525	3.3292	3.3475	3.1236		
3	−3.3143	−1.4619	3.2090	2.9944	3.1366	3.1330	
4	3.7588	2.2970	3.1647	3.2324	3.1317	3.1332	3.1332
5	−4.1560	−1.8590	3.1468	3.0577	3.1340	3.1332	
6	4.5182	2.6593	3.1392	3.1931	3.1327		
7	−4.8535	−2.1943	3.1359	3.0842			
8	5.1671	2.9728	3.1344				
	$\times 10^{-3}$	$\times 10^{-3}$	$\times 10^{-4}$	$\times 10^{-4}$	$\times 10^{-4}$	$\times 10^{-4}$	$\times 10^{-4}$

The half-cycle contributions, T_n, were evaluated with $a_n = (n\pi/100)^{1/2}$ using twelve 2-point Gauss–Legendre formulas. It is noticed in this case that the original sequence $\{A_n\}$ is divergent. The Euler transformation produces the sequence $\{E_n\}$ which is slowly convergent. The sequence $e_1\{A_n\}$ is also seen to be slowly convergent, but when the operator e_1 is used iteratively, the successive sequences $e_1^2\{A_n\}$, $e_1^3\{A_n\}$,... are seen to be converging rapidly. The application of the operator e_2 to the original sequence $\{A_n\}$ produces the sequence $\{3.1268, 3.1354, 3.1322, 3.1337, 3.1329, \ldots \times 10^{-4}\}$ and e_2^2 produces 3.1332×10^{-4} as its first term.

In conclusion, it appears that the Shanks' acceleration technique is a powerful tool for the evaluation of oscillatory integrals with an infinite range. This is particularly true when the oscillations are damped in a predominantly exponential manner or fall off, for example, as $1/x^p$ where p is sufficiently large (say $p \geqslant 1$). In these cases Shanks' technique proves more economical in terms of function evaluations than the well known Euler transformation. However, when the higher order difference terms in the Euler formula are small (for example, when the convergence is very slow or when polynomial behavior is exhibited) the Euler method converges extremely rapidly.

Also, for integrals which converge in the mean only, Shanks' method proves successful, and may again converge more rapidly than Euler's transformation, depending on the behavior of the original sequence.

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